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Contact Transformations and Conformal Group. II. Non-Relativistic Theory

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Abstract

We find the contact group of non-relativistic accelerated systems. A comparison with the relativistic situation is made.

1. *Introduction*

In a previous work (Boya & Cerveró, 1974), the general theory of contact transformations in a 1 + 1 space-time model has been stated in connection with those transformations which conserve systems with constant relative acceleration. The purpose of the present work is to apply this scheme to the non-relativistic situation and to make a comparison between both cases.

In Section 2 we obtain the infinitesimal group for the non-relativistic case, and through the commutation rules we show in Section 3 the equivalence with the contact relativistic group; the corresponding coordinate transformation is highly non-linear. The results of the contraction group theory are applied in Section 4 to connect the punctual subgroups in both cases (hyperbolic conformal group in the relativistic case, and 'Hill group' in the other). The overall relation, including the Galilei group, is completed in Section 5.

2. *Non-Relativistic Contact Transformation Group*

If $(x, t, v = dx/dt)$ are the coordinates in the 'tangent projective bundle' over $\mathbb{R}^2$: (x, t) (cf. Boya & Cerveró, 1974), an infinitesimal contact transformation:

$$\begin{aligned} x &\to x' = x + \tau\eta(x, t, v) \\ t &\to t' = t + \tau\eta_0(x, t, v) \\ v &\to v' = v + \tau\omega(x, t, v) \end{aligned} \tag{2.1}$$

has a generating function

$$\phi = v\eta_0 - \eta \tag{2.2}$$

If $a = d^2x/dt^2$, $b = d^3x/dt^3$, and the extended contact transformations are $a \to a' = a + \tau k(x, t, \nu)$; $b \to b' = b + \tau\theta(x, t, v)$, then:

$$-k = \left(Y^2 + 2aY \frac{\partial}{\partial v} + a^2 \frac{\partial^2}{\partial v^2} + a \frac{\partial}{\partial x}\right)\Phi \tag{2.3}$$

$$-\theta = \left(Y^3 + 3aY^2 \frac{\partial}{\partial v} + 3a^2 Y \frac{\partial^2}{\partial v^2} + a^3 \frac{\partial^3}{\partial v^3} + 3aY \frac{\partial}{\partial x} + 3a^2 \frac{\partial^2}{\partial x\, \partial v}\right)\Phi + b\left(3a \frac{\partial^2}{\partial v^2} + 3Y \frac{\partial}{\partial v} + \frac{\partial}{\partial x}\right)\Phi \tag{2.4}$$

where

$$Y = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}$$

The differential equation of all non-relativistic reference systems with constant relative acceleration is obviously:

$$b = \frac{d^3x}{dt^3} = 0 \tag{2.5}$$

i.e. $\theta = 0$ in (2.4), which gives rise to the following equations, obtained by putting equal to zero each power of a:

$$Y^3\Phi = 0 \tag{2.6}$$

$$Y^2 \frac{\partial \Phi}{\partial v} + Y \frac{\partial \phi}{\partial x} = 0 \tag{2.7}$$

$$Y \frac{\partial^2 \phi}{\partial \nu^2} + \frac{\partial^2 \phi}{\partial \nu\, \partial x} = 0 \tag{2.8}$$

$$\frac{\partial^3 \phi}{\partial \nu^3} = 0 \tag{2.9}$$

The last equation has a general solution:

$$\Phi(x, t, \nu) = A(x, t)\nu^2 + B(x, t)\nu + C(x, t) \tag{2.10}$$

Equations (2.6) to (2.8) then imply:

$$A_x = 0$$
$$A_t = -\tfrac{1}{2}B_x$$
$$B_{xx} = 0$$
$$B_{tt} = -C_{tx}$$
$$2A_{tt} + 3B_{tx} + C_{xx} = 0 \tag{2.11}$$

'All third-order derivatives of A, B, C are zero' (2.12)

which allows a complete solution for the generating function in terms of ten arbitrary constants:

$$\begin{aligned}\Phi = {} & \lambda_1(\tfrac{1}{2}\nu^2t^2 - \upsilon tx + x^2) + \lambda_2(\tfrac{1}{2}\nu^2t - \nu x) \\ & + \lambda_3(\tfrac{1}{2}\nu^2) + \lambda_4(\nu t + x) + \lambda_5(\nu t - x) + \lambda_6 t \\ & + \lambda_7(\tfrac{1}{2}t^2) + \lambda_8(\tfrac{1}{2}\nu t^2 - tx) + \lambda_9\nu + \lambda_{10}\end{aligned} \tag{2.13}$$

and from the known relations between Φ and the infinitesimal contact group:

$$\eta_0 = \frac{\partial\phi}{\partial\nu}; \qquad \eta = \nu\frac{\partial\phi}{\partial\nu} - \phi; \qquad \omega = -Y\phi$$

We can write ten independent invariant vector fields in (x, t, ν) in the form:

$$Y_1 = (\tfrac{1}{2}\upsilon t^2 - tx)\frac{\partial}{\partial t} + (\tfrac{1}{4}\nu^2t^2 - x^2)\frac{\partial}{\partial x} + (\tfrac{1}{2}\nu^2t - \nu x)\frac{\partial}{\partial\nu}$$

$$Y_2 = (\nu t - x)\frac{\partial}{\partial t} + \tfrac{1}{2}\nu^2t\frac{\partial}{\partial x} + \tfrac{1}{2}\nu^2\frac{\partial}{\partial\nu}$$

$$Y_3 = \nu\frac{\partial}{\partial t} + \tfrac{1}{2}\nu^2\frac{\partial}{\partial x}$$

$$Y_4 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - 2\nu\frac{\partial}{\partial\nu}$$

$$Y_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$$

$$Y_6 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial\nu}$$

$$Y_7 = \tfrac{1}{2}t^2 \frac{\partial}{\partial x} + t \frac{\partial}{\partial \nu}$$

$$Y_8 = \tfrac{1}{2}t^2 \frac{\partial}{\partial t} + xt \frac{\partial}{\partial x} + x \frac{\partial}{\partial \nu}$$

$$Y_9 = \frac{\partial}{\partial t}$$

$$Y_{10} = \frac{\partial}{\partial x} \tag{2.14}$$

The corresponding commutation relations can be easily computed; we omit the results for brevity. As in Boya & Cerveró (1974), the linear transformation $\alpha = Y_5$ and $\beta = \frac{1}{2}(Y_5 - Y_4)$:

$$\begin{aligned}
Y_1 &= -\tfrac{1}{2}(J_{34} + J_{35} + J_{14} + J_{15}) \\
Y_3 &= -\tfrac{1}{2}(J_{34} + J_{35} - J_{14} - J_{15}) \\
Y_7 &= -\tfrac{1}{2}(J_{34} - J_{35} + J_{14} - J_{15}) \\
Y_{10} &= -\tfrac{1}{2}(J_{34} - J_{35} - J_{14} + J_{15}) \\
Y_8 &= -\frac{1}{\sqrt{2}}(J_{12} + J_{32}), \qquad Y_9 = -\frac{1}{\sqrt{2}}(J_{12} - J_{32}) \\
Y_2 &= -\frac{1}{\sqrt{2}}(J_{24} + J_{25}), \qquad Y_6 = -\frac{1}{\sqrt{2}}(J_{24} - J_{25}) \\
\beta &= -J_{45} \qquad \alpha = J_{31}
\end{aligned} \tag{2.15}$$

allows us to write the commutation relations in compact form as

$$[J_{\alpha\beta}, J_{\gamma\delta}] = (g_{\beta\gamma}J_{\alpha\delta} + g_{\alpha\delta}J_{\beta\gamma} - g_{\alpha\gamma}J_{\beta\delta} - g_{\beta\delta}J_{\alpha\gamma}) \tag{2.16}$$

where $g_{\alpha\beta}$: (diag $g_{\alpha\beta}$: $++-+-$); i.e. the group will be $O_{3,2}$.

3. *Equivalence Between Relativistic and Non-Relativistic Cases*

In Boya & Cerveró (1974) we obtained the same group for the relativity case, in fact, the isomorphism between both cases was already noted by Hill (1945). We expect a non-linear transformation in the $\mathbb{R}^3$ spaces, which brings about that equivalence, in fact, if

$$\left.\begin{aligned}
x_0' &= -2i(x + x_0)\sqrt{\frac{1-p}{1+p}} \\
x' &= i\left[(x - x_0) - (x + x_0)\frac{1-p}{1+p}\right] \\
p' &= \sqrt{\frac{1-p}{1+p}}
\end{aligned}\right\} \tag{3.1}$$

and remembering $x_0 = ct$ and $p = v/c$, a straightforward but tedious calculation gives (cf. Boya & Cerveró, 1974):

$$
\begin{array}{ll}
X_1 \to -\dfrac{i}{2c^2} Y_8 & X_6 \to -\dfrac{i}{c}(Y_1 + \tfrac{1}{2}Y_7) \\[2ex]
X_2 \to -\dfrac{i}{c^2}(Y_2 + Y_6) & X_7 \to Y_5 \\[2ex]
X_3 \to \dfrac{i}{c}(Y_2 - \tfrac{1}{2}Y_6) & X_8 \to -\dfrac{1}{2c}(Y_5 - Y_4) \\[2ex]
X_4 \to -\dfrac{i}{c^2} Y_9 & X_9 \to -i(Y_{10} + 2Y_3) \\[2ex]
X_5 \to -i(Y_1 - \tfrac{1}{2}Y_7) & X_{10} \to \dfrac{i}{c}(Y_{10} - 2Y_3)
\end{array}
\tag{3.2}
$$

In this transformation the group structure is not changed, as for example, the 'i' affects an *even* number of signs in the metric.

4. *Group Contraction*

The Galilei group appears, as is well known, from contraction of the Poincaré group (Inönü & Wigner, 1953, 1954; Inönü, 1964). Does this relation persist for the conformal and contact conformal extensions? This is the question we address ourselves here. Of course, in the last step we have isomorphism; thus no contraction is allowed. But the conformal hyperbolic group (written with $x_0 = ct$):

$$X_5 = \frac{c^2t^2 + x^2}{2c^2}\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + x\left(1 - \frac{v^2}{c^2}\right)\frac{\partial}{\partial v}$$

$$X_6 = \frac{xt}{c^2}\frac{\partial}{\partial t} + \frac{c^2t^2 + x^2}{2c^2}\frac{\partial}{\partial x} + t\left(1 - \frac{v^2}{c^2}\right)\frac{\partial}{\partial v}$$

$$X_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$$

$$X_8 = \frac{x}{c^2}\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial}{\partial v}$$

$$X_9 = \frac{\partial}{\partial t}; \qquad X_{10} = \frac{\partial}{\partial x} \tag{4.1}$$

becomes, when $c \to \infty$:

$$X_5 \to X_5{}^c = \tfrac{1}{2}t^2 \frac{\partial}{\partial t} + xt \frac{\partial}{\partial x} + x \frac{\partial}{\partial \nu} = Y_8$$

$$X_6{}^c = \tfrac{1}{2}t^2 \frac{\partial}{\partial x} + t \frac{\partial}{\partial \nu} = Y_7$$

$$X_7{}^c = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} = Y_5, \qquad X_9{}^c = \frac{\partial}{\partial t} = Y_9$$

$$X_8{}^c = t \frac{\partial}{\partial x} + \frac{\partial}{\partial \nu} = Y_6, \qquad X_{10}{}^c = \frac{\partial}{\partial x} = Y_{10} \tag{4.2}$$

This is equivalent to a contraction with respect to the subgroup generated by X_5, X_7 and X_9.

One thus expects to obtain the point group of the non-relativistic equation (2.5), but this is not so because that group includes $Y_5, \ldots, Y_{10}$ *plus* Y_4, and Y_4 is not obtained from contraction. In other words, the transformation (2.2) does not preserve the projection: contact group $\to$ point group.

5. *Galilei Group and Conformal Transformations*

As in Kastrup (1966), it is to possible to prove that the contact group of

$$a_0 = \frac{d^2x}{dt^2} = \text{cons.} \tag{5.1}$$

is a point group (including the Galilei group), unenlarged by pure contact transformations. In fact from (2.3), (5.1) implies $k = a_0'$, that is a system of equations, one of which is

$$\frac{\partial^2 \phi}{\partial \nu^2} = 0 \tag{5.2}$$

which, as remarked in Boya & Cerveró (1974), imply pure point transformations (i.e. η and η_0 do not depend on ν).

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